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NOTE

COMBINATORIAL ORTHOGONALITY IN ABSTRACT CONFIGURATIONS

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Abstract. In the linear space of real valued functions on the power set of a finite set, the orthogonal complement of the subspace spanned by the characteristic functions of the maximal chains is precisely the subspace spanned by the functions $\mu_i - \mu_0$, where μ_i denotes the characteristic function of the ith rank. In this paper it is shown that in more general abstract configurations than power sets, this orthogonality relation is equivalent to a certain connectedness property. As a corollary, the relations obtaining between pairs of maximum sized antichains are characterized.

1. Introduction

By a *configuration* we mean a partial order of finite length which satisfies the Jordan–Dedekind chain condition [1], such that all maximal chains are of the same length. On a configuration we can always introduce a height function, and partition the configuration into *ranks*, i.e., maximal subsets of constant height. In the linear space E of real valued functions on the configuration P, introduce the inner product $(\alpha, \beta) = \sum_{x \in P} (\alpha x)\beta x$. It is convenient to represent subsets of P by their characteristic functions; thus, with each subset ξ of P, we associate a function (also denoted by ξ) in E defined by

$$\xi x = \begin{cases} 1 & \text{if } x \in \xi\,, \\ 0 & \text{if } x \notin \xi\,. \end{cases}$$

In [3] it is proven that the subspace of E spanned by the maximal chains of P is the orthogonal complement of the subspace spanned by $\mu_1 - \mu_0, \mu_2 - \mu_0, ..., \mu_k - \mu_0$, where $\mu_0, \mu_1, ..., \mu_k$ denote the ranks of P, *provided that P is the power set of a finite set.* In this note we investigate the validity of this characterization for arbitrary configurations P. It will be shown that the characterization remains valid if and only if P enjoys a certain connectivity property. If ξ and ζ are subsets of P, we shall say that ξ *is connected through* ζ provided that for each pair of points u, v of ξ, there exist points $x_0, x_1, ..., x_m$ in $\xi \cup \zeta$, with $x_0 = u$ and $x_m = v$, such that x_i and x_{i-1} are incident for $i = 1, 2, ..., m$.

Theorem. *Let $\mu_0, \mu_1, ..., \mu_k$ denote the ranks of the configuration P. Then for the subspace of E spanned by $\mu_1 - \mu_0, \mu_2 - \mu_0, ..., \mu_k - \mu_0$ to be the orthogonal complement of the subspace spanned by the maximal chains of P, it is necessary and sufficient that μ_{i-1} be connected through μ_i for $i = 1, 2, ..., k$.*

2.

It is clear that $(\gamma, \mu_i - \mu_0) = 0$ for every maximal chain γ, since γ meets each rank exactly once. In this section we prove that the connectivity property is sufficient. Assume therefore that μ_{i-1} is connected through μ_i for $i = 1, 2, ..., k$, and let ξ be orthogonal to every maximal chain. We shall show that ξ is constant on each rank, so that there exist real numbers $a_0, a_1, ..., a_k$ with $\xi = \Sigma a_i \mu_i$. Were this false, let μ_i be the lowest rank on which ξ is not constant. From the orthogonality of ξ to all maximal chains, it follows that $i < k$. Inasmuch as μ_i is connected through μ_{i+1}, for each pair u, v in μ_i, there exist $x_0, x_1, ..., x_m$ in μ_i and $y_1, y_2, ..., y_m$ in μ_{i+1} with $x_0 = u$ and $x_m = v$ such that $x_{j-1} < y_j$ and $x_j < y_j$ for $j = 1, 2, ..., m$. A contradiction will be obtained by proving that $\xi x_{j-1} = \xi x_j$ for $j = 1, 2, ..., m$. Consider a pair γ, δ of maximal chains with γ passing through x_{j-1} and y_j, δ passing through x_j and y_j, and such that $\gamma z = \delta z$ for all $z > y_j$. Since ξ is constant on each rank below μ_i, it follows that $(\xi, \gamma - \delta) = \xi x_{j-1} - \xi x_j$; but $(\xi, \gamma) = (\xi, \delta) = 0$, so $\xi x_{j-1} = \xi x_j$. Now from $\Sigma a_i \mu_i = \xi$ and from the orthogonality of ξ to maximal chains, it follows that $\Sigma a_i = 0$, so that ξ can be written as $\Sigma a_i(\mu_i - \mu_0)$.

3.

To prove necessity, suppose that some rank μ_{i-1} is not connected through μ_i. Let σ be a connected (through μ_i) component of μ_{i-1} and let $\psi = \mu_{i-1} - \nu$. Let λ and π consist of the points of μ_i which cover points of ν and ψ, respectively. Since ν is a component of μ_{i-1}, it is clear that λ and π are disjoint. Then the function $\nu - \lambda$ is orthogonal to every maximal chain, yet is not constant on the rank μ_{i-1}; it is therefore not in the subspace spanned by $\mu_1 - \mu_0, \mu_2 - \mu_0, \ldots, \mu_k - \mu_0$.

4.

Our theorem establishes a necessary and sufficient condition of a *local* character for a *global* relation in E. It is closely related to the uniqueness theorem of Drake [2]. In the corollary (which generalizes [3, Corollary 4]), we shall refer to the necessary and sufficient condition as the *connectivity condition*.

Corollary. *Let P be a configuration which satisfies the connectivity condition, and let ξ and ζ be subsets of P. Then $(\xi, \gamma) = (\zeta, \gamma)$ for every maximal chain γ if and only if the following three conditions hold:*

(i) *there are exactly as many i with $\mu_i \subset \xi$ as there are with $\mu_i \subset \zeta$,*

(ii) *there are exactly as many i with $\mu_i \cap \xi = \emptyset$ as there are with $\mu_i \cap \zeta = \emptyset$;*

(iii) $\mu_i \neq \mu_i \cap \xi \neq \emptyset$ *implies* $\mu_i \cap \xi = \mu_i \cap \zeta$.

References

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[3] D. Lubell, Orthogonal structures on finite partial orders, J. Combin. Theory, to appear.